iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Single-loop divergences in six dimensions

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1977 J. Phys. A: Math. Gen. 10 L63

(http://iopscience.iop.org/0305-4470/10/4/003)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 30/05/2010 at 13:54

Please note that terms and conditions apply.

J. Phys. A: Math. Gen., Vol. 10, No. 4, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

Single-loop divergences in six dimensions

J S Dowker
Department of Theoretical Physics, University of Manchester, Manchester M13 9PL, UK

Received 14 February 1977

Abstract. The leading divergences of single-loop scalar and vector effective Lagrangians in a six-dimensional Ricci-flat Riemannian space are evaluated.

Van Nieuwenhuizen and Wu (1977) have recently discussed the leading divergences of pure Einstein quantum gravitation. They considered the general structure of these divergences at the two-loop level in four dimensions and at the one-loop level in six dimensions.

It is not without interest to investigate the same questions for the simpler cases of quantum fields of spins zero and one propagating in a background metric. In particular the one-loop divergences in six dimensions can be explicitly evaluated without very much work and this is what I wish to do here, if only to draw attention to a considerable body of interesting and valuable work performed by mathematicians over the past few years.

Van Nieuwenhuizen and Wu use the general ideas of dimensional regularization but this is not important. I shall use what I have termed zeta-function regularization (Dowker and Critchley 1976).

It is shown in the last reference that the singularity in the effective action due to a loop of a scalar particle is of the form

$$W^{(1)}_{\text{pole}} = -\tfrac{1}{2}\mathrm{i} \lim_{\nu\to 1} \zeta(0, m^2)\frac{1}{\nu - 1} \tag{1}$$

where $\zeta(\nu, m^2)$ is the space–time integrated zeta function on the manifold in question.

In general $\zeta(0, m^2)$ is given, in d (even) dimensions by

$$\zeta(0, m^2) = \frac{\mathrm{i}}{(4\pi)^{d/2}} \sum_{n=0}^{d/2} \frac{(-m^2)^{\frac{1}{2}d-n}}{(\frac{1}{2}d-n)!} \int a_n \tag{2}$$

where the a_n are the standard coincidence limits in the Fock–Schwinger–De Witt proper-time expansion. (Mathematicians sometimes call these the Minakshisundaram coefficients.)

Simply in order to compare with dimensional regularization, note that ω (= half the complex dimension) $= \frac{1}{2}d - \nu + 1$, so we have

$$W^{(1)}_{\text{pole}}(d, \text{scalar}) = -\frac{1}{(4\pi)^{d/2}} \frac{1}{2\omega - d} \sum_{n=0}^{d/2} \frac{(-m^2)^{\frac{1}{2}d-n}}{(\frac{1}{2}d-n)!} \int a_n. \tag{3}$$

This is the complete, formal answer with the a_n given in terms of the curvature.

However, only $a_0(=1)$, a_1, a_2 and a_3 have been evaluated so far. Historically a_1 and a_2 were first derived by De Witt in 1963 (De Witt 1965). Berger (1966) later re-derived a_1, while McKean and Singer (1967) are usually credited by the mathematicians with the evaluation of a_2. The explicit formula for a_3 is due to the work of Sakai (1971). These calculations have been tidied up and extended by Gilkey (1973, 1975a,b). The work on Donnelly (1974, 1975) and Patodi (1970, 1971) must also be mentioned.

Gilkey's theory gives the coefficients for an arbitrary second-rank differential operator, in particular for the covariant Laplacian, and for any vector bundle in the tangent space. For four dimensions his formulae, taken together with equation (3), constitute an extension and a more elegant derivation of the 'algorithm' of 't Hooft and Veltman (1974).

I now restrict the discussion to the massless case so that (3) becomes

$$W^{(1)}_{\text{pole}}(d, \text{massless scalar}) = -\frac{1}{(4\pi)^{d/2}}\frac{1}{2\omega - d}\int a_{d/2} \tag{4}$$

and only $a_{d/2}$ is required. Thus the available expressions for a_n mean that we can investigate the six-dimensional case.

According to the results of Sakai, Gilkey and Donnelly, referred to above, the integrated a_3 is, in a Ricci-flat space (as discussed by van Nieuwenhuizen and Wu)

$$\int a_3 = \frac{1}{6!}\int\left(\frac{3}{5}|\nabla R|^2 - \frac{896}{315}Y - \frac{104}{315}X\right)$$

where

$$|\nabla R|^2 = R_{\mu\nu\rho\sigma\|\alpha}R^{\mu\nu\rho\sigma\|\alpha}$$
$$X = R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} = -A_1$$
$$Y = R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}R^{\nu}{}_{\alpha}{}^{\sigma}{}_{\beta} = A_2.$$

Green's theorem gives the integral relation

$$\int |\nabla R|^2 = \int (4Y + X)$$

and there are the topological results

$$\int \left(\tfrac{1}{2}X - Y\right) = \begin{cases} 48\pi^3\chi(M) & d = 6 \\ 0 & d < 6, \end{cases}$$

where $\chi(M)$ is the Euler–Poincaré characteristic for the space–time, M. These relations lead to

$$\int a_3 = \frac{1}{6!}\frac{47}{105}\int X + \frac{4}{135}\pi^3\chi(M)$$

which gives an explicit form for the pole term (4) in six dimensions.

The 'photon' can be treated in a similar fashion. If Maxwell theory is extended to d dimensions in the most 'natural' way, i.e. by letting the indices range over d values, then equation (4) is modified by replacing $a_{d/2}$ by

$$\operatorname{Tr} a_{d/2}(\mathrm{V}) - 2a_{d/2}(\mathrm{S})$$

where $a_{d/2}(\mathrm{V})$ and $a_{d/2}(\mathrm{S})$ are the coefficients for the vector and scalar fields respectively. The (minimal) scalar contribution is a ghost effect.

If the formulae of Gilkey (1975b) are employed I find that, in the Ricci-flat case,

$$\int \mathrm{Tr}\, a_3(\mathrm{V}) = -\frac{373}{12600}\int X + \frac{172}{45}\pi^3\chi(M)$$

and for the total pole numerator,

$$\int (\mathrm{Tr}\, a_3(\mathrm{V}) - 2a_3(\mathrm{S})) = -\frac{577}{18720}\int X + \frac{104}{27}\pi^3\chi(M).$$

Interesting information on the structure of compact Ricci-flat manifolds can be found in the work of Fischer and Wolf (1975).

Since this calculation is somewhat academic in six dimensions I have not bothered to work out the pole term for an arbitrary Riemannian space–time although the evaluation is perfectly straightforward. Instead the method is being extended to pure gravitation so that the coefficient α_6 of van Nieuwenhuizen and Wu (1977) can be determined analytically.

References

Berger M 1966 *C.R. Acad. Sci., Paris* **263** 13

De Witt B S 1965 *Dynamical Theory of Groups and Fields* (New York: Gordon and Breach)

Donnelly H 1974 *Indiana Univ. Math. J.* **24** 603

—— 1975 *Proc. Symp. Pure Math.* vol. 27 (Providence, RI: American Mathematical Society) p 195

Dowker J S and Critchley R 1976 *Phys. Rev.* D **13** 3225

Fischer A E and Wolf J A 1975 *J. Diff. Geom.* **10** 277

Gilkey P B 1973 *Adv. Math.* **10** 344

—— 1975a *Proc. Symp. Pure Math.* vol. 27 (Providence, RI: American Mathematical Society) p 265

—— 1975b *J. Diff. Geom.* **10** 601

't Hooft G and Veltman M 1974 *Ann. Inst. Henri Poincaré* **20** 69

McKean H P and Singer I M 1967 *J. Diff. Geom.* **1** 43

van Nieuwenhuizen P and Wu C C 1977 *J. Math. Phys.* **18** 182

Patodi V K 1970 *J. Indian Math. Soc.* **34** 269

—— 1971 *J. Diff Geom.* **5** 251

Sakai T 1971 *Tohoku Math. J.* **23** 589